The Structure of Bernoulli's Proof of the Law of Large Numbers

Let R be the number of ways of getting a success in a single trial and let S be the number of ways of getting a failure in a single trial. Let T = R + S and let W_i be the number of ways of getting exactly i successes in NT trials. Bernoulli wanted to prove that the ratio

$$\frac{W_{NR-N}+W_{NR-N+1}+\ldots+W_{NR+N}}{W_0+W_1+\ldots+W_{NR-N-1}+W_{NR+N+1}+W_{NR+N+2}\ldots+W_{NT}}$$

could be made as large as desired by making N sufficiently large.

This ratio is the ratio of the number of ways of getting a number of successes in the range NR – N through NR + N to the number of ways of getting a number of successes outside that range.

The proof involves proving that:

1) $$\frac{W_{NR}}{W_{NR+N}}<\frac{W_{NR+1}}{W_{NR+N+1}}<\ldots<\frac{W_{NR+N}}{W_{NR+2N}}$$

from 1) Bernoulli got:

2) $$\frac{W_{NR}}{W_{NR+N}} < \frac{W_{NR+1}+W_{NR+2}+\ldots+W_{NR+N}}{W_{NR+N+1}+W_{NR+N+2}+\ldots+W_{NR+2N}}$$

and because there were (S – 1)N terms from W_{NR+N+1} to W_{NT} and because Bernoulli knew the terms were decreasing because their subscripts were greater than NR he got:
$(S-1)(W_{NR+N+1}+W_{NR+N+2}+\ldots+W_{NR+2N}) > (W_{NR+N+1}+\ldots+W_{NT})$
and from this inequality he got:

3) $$\frac{W_{NR+1}+W_{NR+2}+\ldots+W_{NR+N}}{(s-1)\times(W_{NR+N+1}+W_{NR+N+2}+\ldots+W_{NR+2N})} < \frac{W_{NR+1}+W_{NR+2}+\ldots+W_{NR+N}}{W_{NR+N+1}+W_{NR+N+2}+\ldots+W_{NT}}$$

So if W_{NR}/W_{NR+N} could be made as large as desired by making N sufficiently large, then the right side of 2) could be made as large as desired and then the left side of 3) could also be made as large as desired and finally the right side of 3):

$$\frac{W_{NR+1}+W_{NR+2}+\ldots+W_{NR+N}}{W_{NR+N+1}+W_{NR+N+2}+\ldots+W_{NT}}$$

could also be made as large as desired by making N sufficiently large.

This would give him half of the formula he needed.

The other half of the formula involved the W's whose subscripts were less than NR and using the same methods as above it could be shown that if W_{NR}/W_{NR-N} could be made as large as desired by making N sufficiently large that:

$$\frac{W_{NR-1}+W_{NR-2}+\ldots+W_{NR-N}}{W_{NR-N-1}+W_{NR-N-2}+\ldots+W_{0}}$$

could also be made as large as desired by making N sufficiently large. Bernoulli concluded that the ratio of the sum of the numerators of this result and the previous result to the sum of the denominators of this result and the previous result could also be made as large as desired by making N sufficiently large. This ratio was the same as the ratio he needed except that W_{NR} was missing from the numerator which would make the ratio even larger if it were added.

So all it remained to do was to prove that W_{NR}/W_{NR+N} and W_{NR}/W_{NR-N} could both be made as large as desired by making N sufficiently large. The demonstrations that

Bernoulli gave in his lemma 4 were unsatisfactory and the correct demonstrations were given in his explanatory comment (scholium).

Bernoulli used the expansion of $(R + S)^{NT}$ in his proof but it wasn't necessary. All be needed was the probability formula:

$$P(K)=\frac{(NT)\times(NT-1)\times...\times(NT-K+1)\times R^{K}\times S^{NT-K}}{1\times 2\times...\times K\times T^{K}\times T^{NT-K}}$$

Bernoulli noted near the end of his proof that if the T's in the denominator of this formula were omitted it gave the number of ways of getting K successes in NT trials and that the terms in the expansion of $(R + S)^{NT}$ were the same as the number of ways.
Notice that in the denominator of the formula, Bernoulli has 1x2 x...x K.

In the formula I used, I had Kx(K-1)x...x1. This explains why my formula for Bernoulli's M/L(my W_{NR}/W_{NR+N}) has the terms in the numerator in the reverse order than Bernoulli had. This made it possible for me to come up with a different proof that M/L could be made as large as desired by making N sufficiently large.

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